Towards Instance-Optimal Private Query Release

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Abstract

We study efficient mechanisms for the query release problem in differential privacy: given a workload of \(m\) statistical queries, output approximate answers to the queries while satisfying the constraints of differential privacy. In particular, we are interested in mechanisms that optimally adapt to the given workload. Building on the projection mechanism of Nikolov, Talwar, and Zhang, and using the ideas behind Dudley’s chaining inequality, we propose new efficient algorithms for the query release problem, and prove that they achieve optimal sample complexity for the given workload (up to constant factors, in certain parameter regimes) with respect to the class of mechanisms that satisfy concentrated differential privacy. We also give variants of our algorithms that satisfy local differential privacy, and prove that they also achieve optimal sample complexity among all local sequentially interactive private mechanisms.

1 Introduction

There is an inherent tension in data analysis between privacy and statistical utility. This tension is captured by the Fundamental Law of Information Recovery: Revealing “overly accurate answers to too many questions will destroy privacy”. This tension, however, is not equally pronounced for every set of queries an analyst may wish to evaluate on a sensitive dataset. As a simple illustration, a single query repeated \(m\) times is much easier to answer while preserving privacy than is a collection of \(m\) random queries. For this reason, one of the basic goals of algorithmic privacy research is to design efficient private algorithms that optimally adapt to the structure of any given collection of queries.

Phrased more specifically, this goal is to design algorithms achieving nearly optimal sample complexity—the minimum dataset size required to privately produce answers to within a prescribed accuracy—for any given workload of queries.

In this work, we address this problem in the context of answering statistical queries in both the central and local models of differential privacy. Building on the projection mechanism [NTZ10], and using the ideas behind Dudley’s chaining inequality, we propose new algorithms for privately answering statistical queries. Our algorithms are efficient and achieve instance optimal sample complexity (up to constant factors, in certain parameter regimes) with respect to a large class of differentially private algorithms. Specifically, for every collection of statistical queries, our algorithms provide answers with constant average squared error with the minimum dataset size requirement amongst all algorithms satisfying concentrated differential privacy (CDP) [DR16 BS16]. We further show that our algorithmic techniques can be adapted to work in the local model of differential privacy, where they again achieve optimal sample complexity amongst all algorithms with constant average squared error.

1.1 Background

A dataset \(X\) is a multiset of \(n\) elements from a data universe \(\mathcal{X}\). A statistical (also referred to in the literature as “linear”) query is specified by a function \(q : X \to [0, 1]\). Overloading notation, the value of the query \(q\) on a dataset \(X\) is

\[
q(X) = \frac{1}{n} \sum_{x \in X} q(x) \in [0, 1],
\]

where dataset elements appear in the sum with their multiplicity. A query workload \(Q\) is simply a set of \(m\) statistical queries. We use the notation \(Q(X) = (q(X))_{q \in Q} \in [0, 1]^m\) for the vector of answers to the queries in \(Q\) on dataset \(X\).

In the centralized setting in which the dataset \(X\) is held by a single trusted curator, we model privacy by (zero-)concentrated differential privacy. This definition was introduced by Bun and Steinke [BS10], and is essentially equivalent to the original definition of (mean-)concentrated differential privacy proposed by Dwork and Rothblum [DR16], and closely related to Mironov’s...

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This formulation is from [DR14], while the quantitative version is from [DN03].
Rényi differential privacy [Mir17]. Before we state the definition, we recall that two datasets \( X, X' \) of size \( n \) are neighboring if we can obtain \( X' \) from \( X \) by replacing only one of its elements with another element of the universe \( X \).

**Definition 1.** A randomized algorithm \( A \) satisfies \( \rho \)-\( zCDP \) if for any two neighboring datasets \( X \) and \( X' \), and all \( \gamma \in (1, \infty) \),

\[
D_\gamma(A(X) \| A(X')) \leq \gamma \rho,
\]

where \( D_\gamma \) denotes the Rényi divergence of order \( \gamma \) measured in nats.

For the definition of Rényi divergence and further discussion of concentrated differential privacy, we refer the reader to Section 2.3. For now, we remark that concentrated differential privacy is intermediate in strength between “pure” (\( \varepsilon \)-) and “approximate” ((\( \varepsilon, \delta \))-differential privacy, in the sense that every mechanism satisfying \( \varepsilon \)-differential privacy also satisfies \( \varepsilon^2 \)-\( zCDP \), and every mechanism satisfying \( \rho \)-\( pCDP \) also satisfies \( (\rho + 2\sqrt{\rho \log(1/\delta)}, \delta) \)-differential privacy for every \( \delta > 0 \) (see [BBS16]). Our privacy-preserving techniques (i.e., Gaussian noise addition) give privacy guarantees which are most precisely captured by concentrated differential privacy. In general, concentrated differential privacy captures a rich subclass (arguably, the vast majority) of the techniques in the differential privacy literature, including Laplace and Gaussian noise addition, the exponential mechanism [MT07], sparse vector [DR14], and private multiplicative weights [HR10]. Crucially, concentrated differential privacy admits a simple and tight optimal composition theorem which matches the guarantees of the so-called “advanced composition” theorem for approximate differential privacy [DRY10]. Because of these properties, concentrated differential privacy and its variants has been adopted in a number of recent works on private machine learning, for example [ACG+16, PFCW17, Lee17, PSM+18].

We also study the local model of differential privacy, in which the sensitive dataset \( X \) is no longer held by a single trusted curator, but is instead distributed between \( n \) parties where each party holds a single element \( x_i \). The parties engage in an interactive protocol with a potentially untrusted server, whose goal is to learn approximate answers to the queries in \( Q \). Each party is responsible for protecting her own privacy, in the sense that the joint view of every party except \( i \) should be (\( \varepsilon, \delta \))-differentially private with respect to the input \( x_i \). (See Section 2.3 for the precise details of the definition.) Almost all industrial deployments of differential privacy, including those at Google [EPK14], Apple [APP], and Microsoft [DKY17], operate in the local model, and it has been the subject of intense study over the past few years [BS15, BNSGT17, DJW18]. While the local model of privacy offers stronger guarantees to individuals, it is more restrictive in terms of the available privacy preserving techniques. In particular, algorithms based on the exponential mechanism in general cannot be implemented in the local model [KLN+11]. Nevertheless, we show that our algorithms can be relatively easily adapted to the local model with guarantees analogous to the ones we get in the centralized model. We believe this is evidence for the flexibility of our approach.

In order to discuss error guarantees for private algorithms, let us first introduce some notation. We consider two natural measures of error: average (or root-mean squared) error, and worst-case error. For an algorithm \( A \) we define its error on a query workload \( Q \) and databases of size \( n \) as follows.

\[
\text{err}^2(Q, A, n) = \max_X \left( \mathbb{E} \left[ \sum_{q \in Q} \frac{(A(X)_q - q(X))^2}{m} \right] \right)^{1/2},
\]

\[
\text{err}^\infty(Q, A, n) = \max_X \mathbb{E} \left[ \max_{q \in Q} |A(X)_q - q(X)| \right],
\]

where each maximum is over all databases \( X \) of size \( n \), \( A(X)_q \) is the answer to query \( q \) given by the algorithm \( A \) on input \( X \), and expectations are taken with respect to the random choices of \( A \). The notation is used analogously in the local model, with an interactive protocol \( \Pi \) in the place of the algorithm \( A \).

**1.2 Main Results** For the rest of this paper we will work with an equivalent formulation of the query release problem which is more natural from the perspective of geometric techniques, and will also ease our notation. For a given workload of \( m \) queries \( Q \), we can define the set \( S_Q \subseteq [0, 1]^m \) by \( S_Q = \{Q(x) : x \in X\} \). We can identify each data universe element \( x \) with the element \( Q(x) \) of \( S_Q \), so we can think of \( X \) as just a multiset of elements of \( S_Q \); the true query answers \( Q(X) \) then just become the mean of the elements in \( X \). This motivates us to introduce the mean point problem: given a dataset \( X \) of \( n \) elements from a (finite) universe \( X' \subseteq [0, 1]^m \), approximate the mean \( \bar{X} = \frac{1}{n} \sum_{x \in X} x \), where, as usual, the dataset elements are enumerated with repetition. We assume that the algorithm is explicitly given the set \( X \). By analogy with the query release problem, we can
define measures of error for any given dataset \( X \) by

\[
\begin{align*}
\text{err}^2(X, A) &= \left( \frac{1}{m} \| A(X) - \bar{X} \|_2^2 \right)^{1/2}, \\
\text{err}^\infty(X, A) &= \mathbb{E} \| A(X) - \bar{X} \|_\infty.
\end{align*}
\]

Similarly, we can define for any finite universe \( \mathcal{X} \) the error measures

\[
\begin{align*}
\text{err}^2(\mathcal{X}, A, n) &= \max_{X \in \mathcal{X}^n} \text{err}^2(X, A), \\
\text{err}^\infty(\mathcal{X}, A, n) &= \max_{X \in \mathcal{X}^n} \text{err}^\infty(X, A).
\end{align*}
\]

Algorithms for the query release problem can be used for the corresponding mean point problem, and vice versa, with the same error and privacy guarantees. Therefore, for the rest of the paper we will focus on the mean point problem with the understanding that analogous results for query release follow immediately.

In this work, we give query release algorithms whose error guarantees naturally adapt to the properties of the queries, or, equivalently, we give algorithms for the mean point problem that adapt to the geometric properties of the set \( \mathcal{X} \). Notice that for any datasets \( X \) and \( X' \) of size \( n \) that differ in a single element, \( \bar{X} - \bar{X}' \in \frac{1}{n}(\mathcal{X} + (-\mathcal{X})) \), where \( \mathcal{X} + (-\mathcal{X}) \) is the set of all pairwise sums of elements in \( \mathcal{X} \) and \( -\mathcal{X} \). Since differential privacy should hide the difference between \( X \) and \( X' \), a private algorithm should not reveal where in the set \( \bar{X} + \frac{1}{n}(\mathcal{X} + (-\mathcal{X})) \) the true mean lies. This suggests that the size of \( \mathcal{X} + (-\mathcal{X}) \), and, relatedly, the size of \( \mathcal{X} \) itself, should control the sample complexity of the mean point problem. However, it is non-trivial to identify the correct notion of “size”, and to design algorithms whose sample complexity adapts to this notion. In this work we adopt separation numbers, which quantify how many well-separated points can be packed in \( \mathcal{X} \), as a measure of the size of \( \mathcal{X} \). More precisely, for any set \( S \subseteq \mathbb{R}^m \) and \( t > 0 \), we define the separation number (a.k.a. packing number) as

\[
\mathcal{P}(S, t) = \sup \{|T| : T \subseteq S, \forall x, y \in T, x \neq y \implies \| x - y \|_2 > t\sqrt{m} \}.
\]

That is, \( \mathcal{P}(S, t) \) is the size of the largest set of points in \( S \) whose normalized pairwise distances are all greater than \( t \). Analogously, we define the \( \ell_\infty \) separation number as

\[
\mathcal{P}_\infty(S, t) = \sup \{|T| : T \subseteq S, \forall x, y \in T, x \neq y \implies \| x - y \|_\infty > t \}.
\]

Our bounds for average error will be expressed in terms of \( \mathcal{P}(\mathcal{X}, t) \), and the bounds for worst case error will be expressed in terms of \( \mathcal{P}_\infty(\mathcal{X}, t) \). We will give algorithms whose sample complexity is controlled by the separation numbers, and we will also prove nearly matching lower bounds in the regime of constant error.

### 1.2.1 Average Error

We propose two new algorithms for private query release: the Coarse Projection Mechanism and the Chaining Mechanism. Both algorithms refine the Projection Mechanism of Nikolov, Talwar, and Zhang [NTZ10]. Recall that the Projection Mechanism simply adds sufficient Gaussian noise to \( \bar{X} \) in order to guarantee differential privacy, and then projects the noisy answer vector back onto the convex hull of \( \mathcal{X} \), i.e. outputs the closest point to the noisy answer vector in the convex hull of \( \mathcal{X} \) with respect to the \( \ell_2 \) norm. Miraculously, this projection step can dramatically reduce the error of the original noisy answer vector. The resulting error can be bounded by the Gaussian mean width of \( \mathcal{X} \), which in turn is always at most polylogarithmic in the cardinality of \( \mathcal{X} \).

Our first refined algorithm, which we call the Coarse Projection Mechanism, instead projects onto a minimal \( O(\alpha) \)-cover of \( \mathcal{X} \), i.e. a set \( T \subseteq \mathcal{X} \) such that \( T + O(\alpha \sqrt{m}) \cdot B_m^2 \) contains \( \mathcal{X} \). (Here, “+” denotes the Minkowski sum, i.e. the set of all pairwise sums, and \( B_m^2 \) is the unit Euclidean ball in \( \mathbb{R}^m \).) Since this cover is potentially a much smaller set than \( \mathcal{X} \), the projection may incur less error this way. The size of a minimal cover is closely related to the separation number, and the separation numbers themselves are related to the Gaussian mean width by Dudley’s chaining inequality. We use these connections in the analysis of our algorithm. The guarantees of the mechanism are captured by the next theorem.

**Theorem 1.1.** There exists a constant \( C \) and a \( \rho \)-zCDP algorithm \( \mathcal{A}_{\text{CPM}} \) (the Coarse Projection Mechanism) for the mean point problem, which for any finite \( \mathcal{X} \subseteq [0, 1]^m \), achieves average error \( \text{err}^2(\mathcal{X}, \mathcal{A}_{\text{CPM}}, n) \leq \alpha \) as long as

\[
(1.1) \quad n = \Omega \left( \frac{\log(1/\alpha)}{\alpha^2 \sqrt{\rho}} \cdot H_1(\mathcal{X}, \alpha/C) \right),
\]

where

\[
H_1(\mathcal{X}, \gamma) = \sup \left\{ t \cdot \sqrt{\log(\mathcal{P}(\mathcal{X}, t))} : t \geq \gamma \right\}.
\]

Moreover, \( \mathcal{A}_{\text{CPM}} \) runs in time \( \text{poly}(|\mathcal{X}|, n) \).

We note that the sample complexity of the Coarse Projection Mechanism can be much lower than that of the Projection Mechanism. For example, consider a set \( \mathcal{X} \) defined as a circular cone with apex at the origin, and
a ball of radius $\alpha\sqrt{m}$ centered at $(1 - \alpha)\sqrt{me}_1$ as its base. (Here $e_1$ is the first standard basis vector.) Then, a direct calculation reveals that in order to achieve average error $\alpha$, the Projection Mechanism requires a dataset of size at least $\Omega(\alpha^{-1}/\sqrt{m})$. By contrast, the Coarse Projection Mechanism would project onto the line segment from the apex to the center of the base and achieve error $\alpha$ with a dataset of size $O(\alpha^{-1})$. While in this example $\mathcal{X}$ is not finite, it can be discretized to a finite set without significantly changing the sample complexity.

Inspired by the proof of Dudley’s inequality, we give an alternative Chaining Mechanism whose error guarantees are incomparable to the Coarse Projection Mechanism. Instead of just taking a single cover of $\mathcal{X}$, we take a sequence of progressively finer covers $T_1, \ldots, T_k$. This allows us to write $\mathcal{X}$ as the Minkowski sum $\mathcal{X}^1 + \ldots + \mathcal{X}^k + O(\alpha\sqrt{m})B_{2m}^2$, where the diameter of $\mathcal{X}^i$ decreases with $i$, while its cardinality grows. We can then decompose the mean point problem over $\mathcal{X}$ into a sequence of $k$ mean point problems, which we solve individually with the projection mechanism. The next theorem captures the guarantees of this mechanism.

**Theorem 1.2.** There exists a constant $C$ and a $\rho$-zCDP algorithm $\mathcal{A}_{CM}$ (the Chaining Mechanism) for the mean point problem, which for any finite $\mathcal{X} \subseteq [0,1]^m$ achieves average error $\text{err}^2(\mathcal{X}, \mathcal{A}, n) \leq \alpha$ as long as

$$ n = \Omega \left( \frac{\log(1/\alpha)^{5/2}}{\alpha^2 \sqrt{\rho}} \cdot H_2(\mathcal{X}, \alpha/C) \right), $$

where

$$ H_2(\mathcal{X}, \gamma) = \sup \left\{ t^2 : \sqrt{\log(P(\mathcal{X}, t))} : t \geq \gamma \right\}. $$

Moreover, $\mathcal{A}$ runs in time $\text{poly}(|\mathcal{X}|, n)$.

Notice that the sample complexity upper bound \((1.2)\) depends on a larger power of $\log(1/\alpha)$ than the bound \((1.1)\), but replaces $t$ in the supremum with $t^2$. This change can only improve the latter term, as $P(\mathcal{X}, t) = 1$ for any $t \geq 1$.

### 1.2.2 Instance-Optimality

While our algorithms are generic, we show that for constant error, they achieve optimal sample complexity for any given workload of queries. To be more precise about the instance-optimality of our results, we define the sample complexity of answering a query workload $\mathcal{Q}$ with error $\alpha$ under $\rho$-zCDP by

$$ \text{sc}_\rho^2(\mathcal{Q}, \alpha) = \min \{ n : \exists \rho \text{-zCDP } \mathcal{A} \text{ s.t. } \text{err}^2(\mathcal{Q}, \mathcal{A}, n) \leq \alpha \}, $$

$$ \text{sc}_\rho^\infty(\mathcal{Q}, \alpha) = \min \{ n : \exists \rho \text{-zCDP } \mathcal{A} \text{ s.t. } \text{err}^\infty(\mathcal{Q}, \mathcal{A}, n) \leq \alpha \}. $$

In the local model we analogously define $\text{sc}_{\rho,\delta}^2(\mathcal{Q}, \alpha)$ and $\text{sc}_{\rho,\delta}^\infty(\mathcal{Q}, \alpha)$ with the minimum taken over all protocols satisfying $(\varepsilon, \delta)$-local differential privacy.

For context, let us recall the sample complexity in the centralized model of some known algorithms, and how it compares to the best possible sample complexity. For average error, the projection mechanism [NTZ10] can answer any workload $\mathcal{Q}$ with error at most $\alpha$ under $\rho$-zCDP as long as

$$ n = \Omega \left( \frac{\sqrt{\log |\mathcal{X}|}}{\alpha^2 \sqrt{\rho}} \right). $$

It is known that there exist workloads $\mathcal{Q}$ for which this bound on $n$ matches $\text{sc}_\rho(\mathcal{Q}, \alpha)$ up to constant factors. One particularly natural example here is the workload of 2-way marginals on the universe $\mathcal{X} = \{0,1\}^d$, which consists of $m = \binom{d}{2}$ queries [BUV14]. Thus, the sample complexity of private query release with respect to worst-case workloads of any given size is well-understood. However, we know much less about optimal mechanisms and the behavior of $\text{sc}_\rho(\mathcal{Q}, \alpha)$ for specific workloads $\mathcal{Q}$. This behavior can depend strongly on the workload. For example, for the workload of threshold queries $\mathcal{Q} = \{q_t\}_{t=1}^m$ defined on a totally ordered universe $\mathcal{X} = \{1, \ldots, m\}$ by $q_t(x) = 1 \{x < t\}$, we have sample complexity only $\text{sc}_\rho(\mathcal{Q}, \alpha) = O(\frac{1}{\alpha})$. This motivates the following problems:

1. Characterize $\text{sc}_\rho(\mathcal{Q}, \alpha)$ in terms of natural quantities associated with $\mathcal{Q}$.

2. Identify efficient algorithms whose sample complexity on any workload $\mathcal{Q}$ nearly matches the optimal sample complexity $\text{sc}_\rho(\mathcal{Q}, \alpha)$.

We call algorithms with the property in item 2. above *approximately instance-optimal*. Note that it is a priori not clear that there should exist any *efficient* instance optimal algorithms. Here our notion of efficiency is polynomial time in $n$, the number of queries, and the size of the universe $\mathcal{X}$. This is natural, as this is the size of the input to the algorithm, which needs to take a description of the queries in addition to the database. One could wish for a more efficient algorithm when the queries are specified implicitly, for example by a circuit,
but this has been shown to be impossible in general under cryptographic assumptions [DNR+09].

We prove lower bounds showing instance optimality for our algorithms when the error parameter \( \alpha \) is constant. Once again, we state the lower bounds for the mean point problem, rather than the query release problem. The equivalence of the two problems implies that we get the same optimality results for query release as for the mean point problem. To state the results we extend our notation above to the mean point problem, and define

\[
\text{sc}^2_\rho(\mathcal{X}, \alpha) = \min \{ n : \exists \rho-\text{zCDP } \mathcal{A} \text{ s.t.} \]
\[
\text{err}^2(\mathcal{X}, \mathcal{A}, n) \leq \alpha \},
\]

\[
\text{sc}^\infty_\rho(\mathcal{X}, \alpha) = \min \{ n : \exists \rho-\text{zCDP } \mathcal{A} \text{ s.t.} \]
\[
\text{err}^\infty(\mathcal{X}, \mathcal{A}, n) \leq \alpha \},
\]

and, analogously for \( \text{sc}^{2,\text{loc}}_\epsilon(\mathcal{X}, \alpha) \) and \( \text{sc}^{\infty,\text{loc}}_\epsilon(\mathcal{X}, \alpha) \). Building on the packing lower bounds of Bun and Steinke [BS16], we show that separation numbers also provide lower bounds on \( \text{sc}^2_\rho(\mathcal{X}, \alpha) \). The following theorem is proved in the appendix.

**Theorem 1.3.** For any finite \( \mathcal{X} \subseteq [0,1]^m \) and every \( \alpha, \rho > 0 \), we have

\[
(1.3) \quad \text{sc}^2_\rho(\mathcal{X}, \alpha) \geq \Omega \left( \frac{1}{\alpha \sqrt{\rho}} \cdot H_1(\mathcal{X}, 4\alpha) \right),
\]

where

\[
H_1(\mathcal{X}, \gamma) = \sup \left\{ t : \sqrt{\log(\mathcal{P}(\mathcal{X}, t))} : t \geq \gamma \right\}.
\]

Comparing the lower bound \( (1.3) \) with our algorithmic guarantees \( (1.1) \) and \( (1.2) \), we see that the algorithms in Theorems \( 1.1 \) and \( 1.2 \) can achieve error \( \alpha \) on databases of size at most \( O(\frac{1}{\alpha}) \cdot \text{sc}^2_\rho(\mathcal{X}, \alpha/C') \), where \( \tilde{O} \) hides factors polynomial in \( \log(1/\alpha) \) and \( C' \) is a constant. In other words, when the error \( \alpha \) is constant, our mechanisms have sample complexity which is instance-optimal up to constant factors. The constant error regime is practically the most interesting one and is widely studied in the differential privacy literature. It captures the natural problem of identifying the smallest database size on which the mean point problem (resp. the query release problem) can be solved with non-trivial error. In his survey [Vad17], Vadhan asked explicitly for a characterization of the sample complexity of counting queries in the constant error regime under approximate differential privacy (Open Problem 5.25). Our results make a step towards resolving this question by giving a characterization for the rich subclass of algorithms satisfying concentrated differential privacy.

Beyond the constant error regime, proving instance optimality results with tight dependence on the error parameter \( \alpha \) remains a tantalizing open problem. We note that we are not aware of any \( \mathcal{X} \) for which the sample complexity of the Chaining Mechanism is suboptimal by more than a \( (\log(1/\alpha))^{\Omega(1)} \) factor.

### 1.2.3 Worst-Case Error
Using a variant of the chaining mechanism from Theorem 1.2, we get a guarantee for worst-case error as well.

**Theorem 1.4.** There exists constant \( C \), and a \( \rho \)-zCDP algorithm \( \mathcal{A} \) that for any finite \( \mathcal{X} \subseteq [0,1]^m \) achieves \( \text{err}^\infty(\mathcal{X}, \mathcal{A}, n) \leq \alpha \) as long as

\[
(1.4) \quad n = \Omega \left( \frac{\log(m) \log(1/\alpha)^{5/2}}{\alpha^2 \sqrt{\rho}} \cdot H_2(\mathcal{X}, \alpha/C) \right),
\]

where

\[
H_2(\mathcal{X}, \gamma) = \sup \left\{ t^2 : \sqrt{\log(\mathcal{P}(\mathcal{X}, t))} : t \geq \gamma \right\}.
\]

Moreover, \( \mathcal{A} \) runs in time \( \text{poly}(|\mathcal{X}|, n) \).

This result shows the flexibility of the chaining mechanism. The analysis of the coarse projection mechanism relied crucially on Dudley’s inequality, which is tailored to Euclidean space and the Gaussian mean width. There are, in general, no mechanisms with worst-case error guarantees whose sample complexity depends on the Gaussian mean width, so it is unclear how to adapt the coarse projection mechanism to worst-case error. Nevertheless, by incorporating the idea of chaining used in the proof of Dudley’s inequality inside the algorithm itself, we are able to derive an analogous result.

A lower bound analogous to Theorem 1.3 for worst-case error reveals that the sample complexity of the algorithm in Theorem 1.4 on workload \( Q \) with error \( \alpha \) is at most \( \tilde{O}(\frac{\log(m)}{\alpha}) \cdot \text{sc}^{\infty}_\rho(\mathcal{Q}, \alpha/C') \). I.e., we get instance-optimality up to a \( O(\log m) \) factor for constant \( \alpha \).

### 1.2.4 Local Differential Privacy
Illustrating further the flexibility of our techniques, we show that the Coarse Projection Mechanism and the Chaining Mechanism can be adapted to the local model. The protocols we design are non-interactive, with each party sending a single message to the server, and satisfy pure \( \varepsilon \)-local differential privacy. The protocols are in fact very similar to our algorithms in the central model, except that instead of Gaussian noise we use a variant of the local mean estimation algorithm from [DJKW18] to achieve privacy. The other steps in the Coarse Projection and the Chaining Mechanisms are either pre- or post-processing of the data and can be adapted seamlessly to the local model.
Theorem 1.5. There exists a constant \( C \) and a non-interactive \( \varepsilon \)-LDP protocol \( \Pi_{CPM} \) that for any finite \( \mathcal{X} \subseteq [0,1]^m \) achieves average error \( \text{err}^2(\mathcal{X}, \Pi_{CPM}, n) \leq \alpha \) as long as

\[
(1.5) \quad n = \Omega \left( \frac{\log(1/\alpha)^2}{\alpha^4 \varepsilon^2} \cdot H_1(\mathcal{X}, \alpha/C)^2 \right)
\]

Furthermore, there exists a non-interactive \( \varepsilon \)-LDP protocol \( \Pi_{CM} \) that achieves average error \( \text{err}^2(\mathcal{X}, \Pi_{CM}, n) \leq \alpha \) as long as

\[
(1.6) \quad n = \Omega \left( \frac{\log(1/\alpha)^6}{\alpha^4 \varepsilon^2} \cdot H_2(\mathcal{X}, \alpha/C)^2 \right),
\]

where

\[
H_p(\mathcal{X}, \gamma) = \sup \left\{ t^p \cdot \sqrt{\log(P(t))} : t \geq \alpha/C \right\}.
\]

Both protocols run in time \( \text{poly}(|\mathcal{X}|, n) \).

Moreover, for constant average error \( \alpha \), our algorithms achieve instance-optimal sample complexity up to constant factors. This is true even with respect to \((\varepsilon, \delta)\)-LDP algorithms permitting “sequential” interaction between parties (see Section 2.3 for details of the model). The theorem is proved in the appendix using the framework of Bassily and Smith [BS15].

Theorem 1.6. For any finite \( \mathcal{X} \subseteq [0,1]^m \), every \( \alpha > 0 \), and any \( \delta \) satisfying \( 0 < \delta < \frac{\alpha^2 \varepsilon^2}{C \log(|\mathcal{X}|/\varepsilon)} \) for a sufficiently large constant \( C \), we have

\[
(1.7) \quad \text{sc}_{\varepsilon, \delta}^2(\mathcal{X}, \alpha) \geq \Omega \left( \frac{1}{\alpha^2 \varepsilon^2} \cdot H_1(\mathcal{X}, 6\alpha)^2 \right),
\]

where

\[
H_1(\mathcal{X}, \gamma) = \sup \left\{ t \cdot \sqrt{\log(P(t))} : t \geq \gamma \right\}.
\]

It is an interesting open problem to extend these instance optimality results to worst-case error. While the lower bound extends in a straightforward way, our mechanisms do not, as there is no analog of the projection mechanism for worst-case error, and also no analog of the multiplicative weights mechanism in the local model. Moreover, it is known that in the local model packing lower bounds like these in Theorem 1.6 can be exponentially far from the true sample complexity with respect to worst-case error. For instance, Kasiviswanathan et al. [KLN+11] showed that learning parities over the universe \( \mathcal{X} = \{0,1\}^d \) has sample complexity exponential in \( d \) in the local model, and learning parities easily reduces to answering parity queries with small constant worst-case error. At the same time, packing lower bounds can only show a lower bound which is polynomial in \( d \). Thus, worst case error has substantially different behavior from average error in the local model and requires different techniques.

1.3 Related Work Instance-optimal query release was previously studied in a line of work that brought the tools of asymptotic convex geometry to differential privacy [HT10, BDKT12, NTZ16, Nik15, KN17]. However, despite significant effort, completely resolving these questions for approximate differential privacy appears to remain out of reach of current techniques.

The papers [HT10, BDKT12] focus on pure differential privacy, and their results only apply for very small values of \( \alpha \), while here we focus on the regime of constant \( \alpha \). A characterization for pure differential privacy with constant \( \alpha \) is known [Rot11, BS16, Vad17] based on the same geometric quantities considered in this work. When phrased in our language, these works show that for every constant error parameter \( \alpha \), the sample complexity of the mean point problem with pure differential privacy is characterized up to constant factors by the logarithm of an appropriate separation number of the set \( \mathcal{X} \). The sample complexity lower bound follows from a packing argument. Meanwhile, the upper bound is obtained by using the exponential mechanism of McSherry and Talwar [MT07] to identify a point in a minimal cover of \( \mathcal{X} \) which is as close as possible to \( \bar{X} \). Unlike our algorithms, this application of the exponential mechanism runs in time super-polynomial in \( \mathcal{X} \).

While we prove instance-optimality of our algorithms using similar lower bound techniques (i.e., the generalization of packing arguments to CDP from [BS16]), our new algorithms appear to be completely different. There is no known analogue of the exponential mechanism that is tailored to achieve optimal sample complexity for CDP, and our algorithms are instead based on the projection mechanism.

The papers [NTZ16, Nik15] focus on approximate differential privacy, and give results for the entire range of \( \alpha \), but their bounds are loose by factors polynomial in \( \log |\mathcal{X}| \). We avoid such gaps, since for many natural workloads, such as marginal queries, \( |\mathcal{X}| \) is exponential in the other natural parameters of \( \mathcal{Q} \). The recent paper [KN17] is also very closely related to our work, but does not prove tight upper and lower bounds on \( \text{sc}_p(\mathcal{Q}, \alpha) \) for arbitrary \( \mathcal{Q} \).

2 Preliminaries

In this section we define basic notation, state the definitions of concentrated differential privacy and local differential privacy, and state the known algorithms which will serve as building blocks for our own algorithms. We also describe the geometric tools which will be used throughout this paper.
2.1 Notation

We use the notation $A \subseteq B$ to denote the existence of an absolute constant $C$ such that $A \subseteq CB$, where $A$ and $B$ themselves may depend on a number of parameters. Similarly, $A \gtrsim B$ denotes the existence of an absolute constant $C$ such that $A \geq B/C$.

We use $\| \cdot \|_2$ and $\| \cdot \|_\infty$ for the standard $\ell_2$ and $\ell_\infty$ norms. We use $B^m_2 = \{x : \|x\|_2 \leq 1\}$ to denote the unit $\ell_2$ ball in $\mathbb{R}^m$. For two subsets $S, T \subseteq \mathbb{R}^m$, the notation $S + T$ denotes the Minkowski sum, i.e. the set $\{s + t : s \in S, t \in T\}$.

For a real-valued random variable $Z$ we use the notation $\|Z\|_p = (\mathbb{E}|Z|^p)^{1/p}$.

2.2 Concentrated Differential Privacy

Recalling Definition 1, we say that a randomized algorithm $A$ satisfies $\rho$-zCDP if for any two neighboring datasets $X, X'$ and all $\gamma \in (1, \infty)$, we have

$$D_\gamma(\mathcal{A}(X), \mathcal{A}(X')) \leq \gamma \rho.$$ 

Here, $D_\gamma(\cdot | \cdot)$ denotes the Rényi divergence of order $\gamma$. For probability density functions $P, Q : \Omega \rightarrow \mathbb{R}$ with $P$ absolutely continuous with respect to $Q$, this quantity is defined as

$$D_\gamma(P || Q) = \frac{1}{1 - \gamma} \log \left( \int \frac{P(x)^\gamma}{Q(x)^{1-\gamma}} \, dx \right).$$

For two random variables $Y, Z$, the divergence $D_\gamma(Y || Z)$ is defined as the divergence of their probability densities.

One of the crucial properties of CDP is the following tight composition theorem which matches the guarantees of the so-called “advanced composition” theorem for approximate differential privacy [DRV10].

**Lemma 2.1.** (BST19) Assume that the algorithm $A_1(\cdot)$ satisfies $\rho_1$-zCDP, and, for every $y$ in the range of $A_1$, the algorithm $A_2(\cdot, y)$ satisfies $\rho_2$-zCDP. Then the algorithm $A$ defined by $A(X) = A_2(A_1(X))$ satisfies $(\rho_1 + \rho_2)$-zCDP.

We remark that as a special case of Lemma 2.1, one can take $A_2$ to be a 0-zCDP algorithm which does not directly access the sensitive dataset $X$ at all. In this case, the combined algorithm $A$ satisfies $\rho_1$-zCDP, showing that zCDP algorithms can be postprocessed without affecting their privacy guarantees.

Our algorithms are designed by carefully applying two basic building blocks: the Projection and the Private Multiplicative Weights mechanisms. Below we state their guarantees for the mean point problem.

In order to state the error guarantees for the projection mechanism, we need a couple of definitions. First, let us define the support function of a set $S \subseteq \mathbb{R}^m$ on any $x \in \mathbb{R}^m$ by $h_S(x) = \sup_{y \in S} \langle y, x \rangle$, where $\langle \cdot, \cdot \rangle$ is the standard inner product. If $Z \sim N(0, I)$ is a standard Gaussian random variable in $\mathbb{R}^m$, then we define the Gaussian mean width of a set $S \subseteq \mathbb{R}^m$ by $g(S) = Eh_S(\mathcal{Z})$.

**Lemma 2.2.** ([NTZ16]) Let $m \in \mathbb{N}$ and let $\Delta > 0$. There exists a mechanism $\mathcal{A}_{PM}$ (The Projection Mechanism) such that, for every finite set $X \subseteq \Delta \sqrt{m} \cdot B^m_2$,

$$\text{err}^2(\mathcal{X}, \mathcal{A}, n) \leq \left( \frac{\Delta g(\mathcal{X})}{n \sqrt{2m}} \right)^{1/2} \leq \frac{\Delta (\log |\mathcal{X}|)^{1/4}}{(2\rho)^{1/4} \sqrt{n}}.$$ 

Moreover, $\mathcal{A}_{PM}$ runs in time $\text{poly}(|\mathcal{X}|, n)$.

2.3 Local Differential Privacy

In the local model, the private database $X$ is distributed among $n$ parties, each party holding exactly one element of $X$. For convenience, we index the parties by the integers from 1 to $n$, and denote by $x_i$ the element of $X$ held by party $i$. The parties together with a server engage in a protocol $\Pi$ in order to compute some function of the entire database $X$. Here we consider sequentially interactive protocols (with non-interactive ones as a special case), as defined by Duchi, Wainright, and Jordan [DJW18]. The protocol is defined by a collection of randomized algorithms $\Pi_1, \ldots, \Pi_n$. Algorithm $\Pi_i$ takes as input $x_i$ and a message $Y_{i-1}$ received from party $i - 1$, and produces a pair $(Y_i, Z_i)$, where $Y_i$ is sent to party $i + 1$, and $Z_i$ is sent to the server. Parties 1 and $n$ are exceptions: $\Pi_1$ only takes $x_1$ as input, and $\Pi_n$ only produces $Z_n$ as output. Then the server runs a randomized algorithm $A$ on inputs $Z_1, \ldots, Z_n$ to produce the final output of the protocol. We use $\Pi(X)$ to denote the union of all outputs of the algorithms. The running time of the protocol is the total running time of the algorithms $\Pi_1, \ldots, \Pi_n$ and $A$.

Note that a special case of a sequentially interactive protocol is a non-interactive protocol, in which $\Pi_i$ ignores $Y_i$ and only depends on its private input.
Non-interactive protocols roughly capture the randomized response model of Warner [War65], and their study in the context of differential privacy goes back to [DMNS06].

To formulate our privacy definition in the local model, let us recall the notions of max-divergence and approximate max-divergence, defined for any two random variables $X$ and $Y$ on the same probability space by

\[
D_\infty(X \| Y) = \sup_S \frac{\mathbb{P}(X \in S)}{\mathbb{P}(Y \in S)},
\]

\[
D_\infty^\delta(X \| Y) = \sup_S \frac{\mathbb{P}(X \in S) - \delta}{\mathbb{P}(Y \in S)},
\]

where the supremum is over measurable sets $S$ in the support of $X$. With this notation, the standard definition of an $(\varepsilon, \delta)$-differentially private algorithm [DMNS06] is as follows.

**Definition 2.** A randomized algorithm $A$ satisfies $(\varepsilon, \delta)$-differential privacy if for datasets $X$ and $X'$ we have

\[
D_\infty^\delta(A(X) \| A(X')) \leq \varepsilon.
\]

We will need the simple composition theorem for differential privacy. See the book [DR14] for a proof.

**Lemma 2.4.** Assume that the algorithm $A_1(\cdot)$ satisfies $(\varepsilon_1, \delta_1)$-differential privacy, and, for every $y$ in the range of $A_1$, the algorithm $A_2(\cdot, y)$ satisfies $(\varepsilon_2, \delta_2)$-differential privacy. Then the algorithm $A$ defined by $A(X) = A_2(A_1(X))$ satisfies $(\varepsilon_1 + \varepsilon_2, \delta_1 + \delta_2)$-differential privacy.

The privacy definition for a sequentially interactive protocol in the local model we adopt is as follows.

**Definition 3.** A protocol $\Pi$ in the local model satisfies $(\varepsilon, \delta)$-local differential privacy (LDP) if the algorithm $\Pi_1$ satisfies $(\varepsilon, \delta)$-differential privacy with respect to the single element dataset $\{x_1\}$, and, for every $2 \leq i \leq n$ and every $y$ in the range of $\Pi_{i-1}$, the algorithm $\Pi_i(\cdot, y)$ satisfies $(\varepsilon, \delta)$-differential privacy with respect to the single element dataset $\{x_i\}$.

When a protocol satisfies $(\varepsilon, 0)$-LDP, we also say that it satisfies $\varepsilon$-LDP.

We note that while our protocols work in the non-interactive pure LDP model (i.e. $\delta = 0$), our lower bounds work against the larger class of sequentially interactive protocols and approximate LDP (i.e. sufficiently small but nonzero $\delta$).

### 2.4 Packings, Coverings, and Dudley’s Inequality

Recall the definitions of separation numbers given in the Introduction: for a set $S \subseteq \mathbb{R}^m$ and a proximity parameter $t > 0$ we denote

\[
\mathcal{P}(S, t) = \sup \{|T| : T \subseteq S, \forall x, y \in T, x \neq y \implies \|x - y\|_2 > t\sqrt{m}\},
\]

\[
\mathcal{P}_\infty(S, t) = \sup \{|T| : T \subseteq S, \forall x, y \in T, x \neq y \implies \|x - y\|_\infty > t\}.
\]

Note the non-standard scaling of $\mathcal{P}(S, t)$, which we chose because it corresponds better to the definition of average error.

To prove the optimality of our algorithms, we make use of the well-known duality between packings (captured by separation numbers) and coverings. We say that a set $T \subseteq \mathbb{R}^m$ is a $t$-covering of $S$ with respect to metric $d$ if for every $x \in S$ there exists a point $y \in T$ such that $d(x, y) \leq t$. This definition gives rise to the family of covering numbers (in $\ell_2$ or $\ell_\infty$) of a compact set $S \subseteq \mathbb{R}^m$, defined by

\[
\mathcal{N}(S, t) = \inf \{|T| : \forall x \in S \exists y \in T \text{ st. } \|x - y\|_2 \leq t\sqrt{m}\},
\]

\[
\mathcal{N}_\infty(S, t) = \inf \{|T| : \forall x \in S \exists y \in T \text{ st. } \|x - y\|_\infty \leq t\}.
\]

The next lemma relating separation numbers to covering numbers is folklore (see e.g. Chapter 4 of [AAGM15] for a proof).

**Lemma 2.5.** Let $S$ be a compact subset of $\mathbb{R}^m$, and $t > 0$ be a real number. Let $T$ be a maximal subset of $S$ with respect to inclusion s.t. $\forall x, y \in T : \|x - y\|_2 \geq t\sqrt{m}$ (resp. $\|x - y\|_\infty \geq t$). Then $T$ is a $t$-cover of $S$, i.e. for any $x \in S$ there exists a $y \in T$ such that $\|x - y\|_2 \leq t\sqrt{m}$ (resp. $\|x - y\|_\infty \leq t$). This implies

\[
\mathcal{N}(S, t) \leq \mathcal{P}(S, t), \quad \mathcal{N}_\infty(S, t) \leq \mathcal{P}_\infty(S, t).
\]

We will sometimes have to contend explicitly with the sets described above. A set $T \subseteq \mathbb{R}^m$ is called a $t$-separated set with respect to a metric $d$ if for every $x, y \in T$, we have $d(x, y) \geq t$. In what follows, when we discuss $t$-separated sets in the context of the error measure $\text{err}^2$ the underlying metric will be the scaled $\ell_2$ norm $d(x, y) = \frac{1}{\sqrt{m}}\|x - y\|_2$, and in the context of the error measure $\text{err}_\infty$ the underlying metric will be the $\ell_\infty$ norm $d(x, y) = \|x - y\|_\infty$.

Dudley’s Inequality is a tool which allows us to relate the Gaussian mean width of a set $\mathcal{X} \subset \mathbb{R}^n$ with the family of covering numbers of $\mathcal{X}$ at all scales. (Note that the normalization factor of $\sqrt{m}$ appearing in our
definition of separation/covering numbers causes this statement to differ by a factor of $\sqrt{m}$ from its usual formulation.)

**Lemma 2.6.** ([LT91, Chapter 11.1]) For any subset $X \subseteq \mathbb{R}^m$, with diameter $\sqrt{m} \Delta(X)$, we have

$$g(X) \lesssim \sqrt{m} \int_0^{\Delta(X)} \sqrt{\log N(X,t)} \, dt.$$ 

**2.5 Subgaussian Random Variables** We recall the standard definition of a subgaussian random variable.

**Definition 4.** We say that a mean zero random variable $Z \in \mathbb{R}^m$ is $\sigma$-subgaussian, if for every fixed $\theta \in \mathbb{R}^m$, we have

$$\mathbb{E} \exp((Z, \theta)) \leq \exp(\sigma^2 \|\theta\|^2).$$

For an arbitrary random variable $Z \in \mathbb{R}^m$ we say that it is $\sigma$-subgaussian if $Z - \mathbb{E}Z$ is $\sigma$-subgaussian.

We recall some basic facts about subgaussian random variables in the appendix.

**3 Decompositions of the Universe**

In this section we show a simple decomposition lemma (Lemma 3.1) that underlies all of our new algorithms. We begin by identifying an important property common to both error measures $err^2$ and $err^\infty$ which will be essential to Lemma 3.1.

**Definition 5.** (Subadditive Error Measure)

Let $X \subseteq \mathbb{R}^m$ be a finite universe enumerated as $X = \{x_1, x_2, \ldots, x_T\}$. For each $x_i$, consider an arbitrary decomposition $x_i = x_i^1 + x_i^2$, and define $X^{(1)} = \{x_i^1 : 1 \leq i \leq T\}$, $X^{(2)} = \{x_i^2 : 1 \leq i \leq T\}$. This decomposition induces, for any dataset $X \subseteq \mathbb{R}^m$, a pair of datasets $X^1 \in (X^{(1)})^n$, $X^2 \in (X^{(2)})^n$.

We say that an error measure $err(X, A)$ is a subadditive error measure if for every finite universe $X$, every decomposition as above, every dataset $X \subseteq \mathbb{R}^m$ and every pair of algorithms (or local protocols) $A^1, A^2$, we have

$$err(X, A) \leq err(X^1, A^1) + err(X^2, A^2),$$

where the algorithm (resp. local protocol) $A$ is defined by $A(X) = A^1(X^1) + A^2(X^2)$.

Both error measures of interest in this paper are subadditive error measures.

**Claim 1.** Both $err^2$ and $err^\infty$ are subadditive error measures.

The proof of this claim can be found in the appendix. It follows directly from the triangle inequality for $\ell_2$ and $\ell_\infty$ norms respectively.

**Lemma 3.1.** Let $X \subseteq \mathbb{R}^m$ be a subset of a Minkowski sum $X \subseteq X^{(1)} + X^{(2)} + \ldots + X^{(k)}$, and let $\pi_1, \ldots, \pi_k$ be functions, respectively, from $X$ to $X^{(i)}$ such that $x = \sum_{i=1}^k \pi_i(x)$. Consider an arbitrary subadditive error measure $err$. Let $A^1, A^2, \ldots, A^k$ be a sequence of algorithms (respectively protocols in the local model) such that for every $j$,

1. $err(X^i, A^j) \leq \alpha_j$, and
2. $A^j$ satisfies $\rho_j$-zCDP (resp. $\varepsilon_j$-LDP).

Then we can construct a $\rho$-zCDP mechanism (resp. $\varepsilon$-LDP protocol) $A$ with $err(X, A) \leq \sum_{j=1}^k \alpha_j$, where $\rho = \sum_{j=1}^k \rho_j$ (resp. $\varepsilon = \sum_{j=1}^k \varepsilon_j$).

Moreover, the running time of $A$ is bounded by the sum of the running times of $A^1, \ldots, A^k$, and the running times to compute $\pi_1, \ldots, \pi_k$ on $n$ vectors from $X$. If $A^1, \ldots, A^k$ are non-interactive local protocols, then so is $A$.

**Proof.** We first prove the lemma for CDP. For a database $X \subseteq \mathbb{R}^m$, we can consider a sequence of induced databases $X^1, \ldots, X^k$, where $X^j$ is derived from $X$ by applying $\pi_j$ pointwise to each one of its elements. Given a database $X$ we compute independently $A^j(X^j)$ for every $j$, and release $\sum_{j=1}^k A^j(X^j)$.

The privacy of $A$ follows from the composition properties of zCDP (Lemma 2.1), and postprocessing — i.e. by the composition lemma we know that releasing $(A^1(X^1), \ldots, A^k(X^k))$ satisfies $\rho$-zCDP, and by postprocessing $\sum_{j=1}^k A^j(X^j)$ has the same privacy guarantee.

Moreover, the error bound is satisfied by inductively applying subadditivity of the error measure. Indeed, for any specific database $X$, we have $err(X, A) \leq \sum_{j=1}^k err(X^j, A^j)$, and therefore

$$err(X, A) = \max_{X \in \mathbb{R}^m} err(X, A) \leq \sum_{j=1}^k \max_{X^j \in (X^{(j)})^n} err(X^j, A^j) \leq \sum_{j=1}^k err(X^j, A^j).$$

The proof for LDP is analogous: each party $i$, given input $x_i$, for each $j$, $1 \leq j \leq k$ runs the local protocol $A^j$ with input $\pi_j(x_i)$. The protocols can be run in parallel. At the end the server can compute and output
the sum of the outputs of the local protocols. The error analysis is the same as above, and the privacy bound \( \varepsilon \leq \sum_{j=1}^{k} \varepsilon_j \) follows from the simple composition theorem (Lemma 2.4) for (pure) differential privacy.

4 Algorithms for Concentrated Differential Privacy

In this section we define our two new algorithms in the centralized model. In the subsequent section we describe how to adapt them to the local model.

4.1 The Coarse Projection Mechanism

In this section, we prove Theorem 1.1, giving the guarantees of the coarse projection mechanism.

For a finite \( \mathcal{X} \subset \mathbb{R}^m \), let \( \mathcal{X}^{(1)} \subset \mathbb{R}^m \) be an inclusion-maximal \( \frac{x}{2} \)-separated subset of \( \mathcal{X} \) with respect to the metric \( d(x, y) = \frac{1}{2m} \| x - y \|_2 \). Let \( \mathcal{X}^{(2)} = \frac{1}{2} \sqrt{m} B_2^m \).

We claim that \( \mathcal{X} \subset \mathcal{X}^{(1)} + \mathcal{X}^{(2)} \); this follows since, by Lemma 2.5, \( \mathcal{X}^{(1)} \) is a \( \frac{x}{2} \)-cover of \( \mathcal{X} \).

Let \( A^1 : (\mathcal{X}^{(1)}, \rho) \to \mathcal{X}^{(1)} \) be the projection mechanism (as in Lemma 2.2) and let \( A \) be the trivial 0-zCDP mechanism where \( A^1(x) = 0 \) for all \( x \). Note that \( A^2 \) has error \( \text{err}^2(\mathcal{X}^{(2)}, A^2) \leq \frac{x}{2} \).

We now invoke Lemma 3.1 with \( \mathcal{X}^{(1)}, \mathcal{X}^{(2)} \) and \( A^1, A^2 \) as described, and using the (subadditive) error measure \( \text{err}^2 \). As mentioned in the introduction, this gives the following simple mechanism:

1. Round each element \( x \) of the dataset \( X \) to the nearest point \( x^{(1)} \) in the covering \( \mathcal{X}^{(1)} \) to get a rounded dataset \( \hat{X}^{(1)} \).

2. Add enough Gaussian noise to \( \hat{X}^{(1)} \) to preserve \( \rho \)-zCDP; let the resulting noisy vector be \( \hat{Y} \).

3. Output the closest point in \( \text{conv}(X^{(1)}) \) to \( \hat{Y} \) in \( \ell_2 \).

We will use the following lemma to analyze the error incurred by the projection mechanism (corresponding to steps 2. and 3. above).

**Lemma 4.1.** For any \( \alpha > 0 \) and any \( \frac{x}{2} \)-separated set \( S \subset \mathbb{R}^m \) with diameter \( \sqrt{m} \Delta \), we have

\[
g(S) \leq \sqrt{m} \log(\alpha / \Delta) \sup \left\{ t \log P(S, t) : t \geq \alpha / 4 \right\}.
\]

**Proof.** By Dudley’s inequality (Lemma 2.6), we have

\[
g(S) \leq \sqrt{m} \int_0^\Delta \sqrt{\log N(S, t)} \, dt \leq \int_0^{\alpha/4} \sqrt{\log N(S, t)} \, dt + \int_{\alpha/4}^\Delta \sqrt{\log N(S, t)} \, dt.
\]

Now we can bound the summands on the right hand side separately. Note that for \( t < \alpha / 4 \), because \( S \) is \( \frac{x}{2} \)-separated, we have \( N(S, t) = |S| \leq P(S, \alpha / 2) \) — indeed, every covering of radius \( \frac{x}{4} \) has to contain every point of \( S \). Therefore

\[
\int_0^{\alpha/4} \sqrt{\log N(S, t)} \, dt \leq \frac{\alpha}{4} \sqrt{\log P(S, \alpha / 4)}.
\]

On the other hand, for the second summand, we have

\[
\int_{\alpha/4}^\Delta \sqrt{\log N(S, t)} \, dt = \int_{\alpha/4}^\Delta t \log N(S, t) \, dt \leq \sup_{t > \alpha / 4} \left\{ t \log N(S, t) \right\} \int_{\alpha/4}^\Delta dt
\]

\[
\leq \sup_{t > \alpha / 4} \left\{ t \log P(S, t) \right\} \log \left( \frac{\alpha}{4} \right) \frac{\alpha}{4}.
\]

The last inequality follows from the duality between packing and covering numbers (Lemma 2.3). By (4.9) and (4.10), we have a bound

\[
g(S) \leq \sqrt{m} \log \left( \frac{\alpha}{4} \right) \sup_{t \geq \alpha / 4} \left( t \log P(S, t) \right).
\]

This completes the proof.

Using the guarantee of the projection mechanism (Lemma 2.2), Lemma 3.1 shows that

\[
\text{err}^2(\mathcal{X}, A) \leq \text{err}^2(\mathcal{X}^{(1)}, A^1) + \text{err}^2(\mathcal{X}^{(2)}, A^2)
\]

\[
\leq \left( \frac{\Delta g(\mathcal{X}^{(1)})}{n \sqrt{2pm}} \right)^{1/2} + \frac{\alpha}{2}.
\]

Hence, as soon as \( n \geq \Delta g(\mathcal{X}^{(1)}) / \sqrt{2pm} \alpha^2 \), the first term is bounded by \( \alpha / 2 \), and the total error \( \text{err}^2(\mathcal{A}, \mathcal{X}) \leq \alpha \). By applying Lemma 4.1, we can deduce that it is enough to have

\[
n \geq \frac{\Delta g(\mathcal{X}^{(1)})}{\sqrt{2pm} \alpha^2} \sup \left\{ t \log P(\mathcal{X}^{(1)}, t) : t \geq \alpha / 4 \right\}.
\]

Since \( \mathcal{X}^{(1)} \subset \mathcal{X} \), we have \( P(\mathcal{X}^{(1)}, t) \leq P(\mathcal{X}, t) \) for each \( t \). Therefore, as soon as

\[
n \geq \frac{\Delta g(\mathcal{X}^{(1)})}{\sqrt{2pm} \alpha^2} \sup \left\{ t \log P(\mathcal{X}, t) : t \geq \alpha / 4 \right\}
\]

we have \( \text{err}^2(\mathcal{A}, \mathcal{X}) \leq \alpha \). Finally, because \( \mathcal{X} \subset [0, 1]^m \) in the statement of Theorem 1.1, the diameter of \( \mathcal{X} \) is bounded by \( \sqrt{m} \), we can take \( \Delta = 1 \). This concludes the proof of Theorem 1.1.


4.2 The Chaining Mechanism for Average Error

In this section we will prove Theorem 1.2. The main ingredient of the proof is following decomposition lemma for a finite set $\mathcal{X} \subset [0,1]^m \subset \sqrt{m}B_2$. This decomposition will be used together with Lemma 3.1 to get the desired algorithm.

**Lemma 4.2.** Let $\| \cdot \|$ be a norm on $\mathbb{R}^m$ and let $B = \{ x \in \mathbb{R}^m : \| x \| \leq 1 \}$ be its unit ball. For $\Delta \geq 0$, $\alpha < 1$, and an arbitrary set $\mathcal{X} \subset DB$, there exist a sequence of subsets $\mathcal{X}^{(1)}, \ldots, \mathcal{X}^{(k)}$, with $k = \lceil \log \frac{2}{\alpha} \rceil$, such that $\mathcal{X} \subset \mathcal{X}^{(1)} + \cdots + \mathcal{X}^{(k)} + \frac{\alpha}{2} \Delta B$, where $\mathcal{X}^{(i)} \subset 2^{-i+1} \Delta B$, and $\mathcal{X}^{(i)}$ is $2^{-i}$-separated with respect to the metric $d(x,y) = \| x - y \| / \Delta$. Moreover, $\mathcal{X}^{(1)}, \ldots, \mathcal{X}^{(k)}$ can be computed in time polynomial in $m$ and $|\mathcal{X}|$ if $\| \cdot \|$ can be computed in time polynomial in $m$.

**Proof.** Let $S_1, S_2, \ldots, S_k \subset \mathcal{X}$ be a sequence of subsets such that each $S_j$ is a maximal $2^{-j}$-separated set in the metric $d(x,y)$ defined above. By the maximality of each set, we have

$$\forall x \in \mathcal{X}, \exists y \in S_j \text{ s.t. } \| x - y \| \leq 2^{-j} \Delta.$$  

(4.11)

Let us define a mapping $\pi_j : \mathcal{X} \rightarrow S_j$ such that for each $x$, we have

$$\| \pi_j(x) - x \| \leq 2^{-j} \Delta.$$  

(4.12)

Finally, we define $\mathcal{X}^{(1)} = S_1$, and for $j > 1$, we will take $\mathcal{X}^{(j)} := \{ x - \pi_{j-1}(x) : x \in S_j \}$.

We first observe that $\mathcal{X} \subset \mathcal{X}^{(1)} + \mathcal{X}^{(2)} + \cdots + \mathcal{X}^{(k)} + 2^{-k} \Delta B$. Indeed, for any $x \in \mathcal{X}$, we can take $x_k = \pi_k(x)$, and $x_j = \pi_j(x_{j+1})$ for every $j < k$. Then for every $j < k$ we have $x_{j+1} - x_j \in \mathcal{X}^{(j+1)}$, and we have the telescoping decomposition

$$x = x_1 + \sum_{j=1}^{k-1} (x_{j+1} - x_j) + (x - x_k).$$

Finally, by definition of $x_k$, we have $\| x - x_k \| = \| x - \pi_k(x) \| \leq 2^{-k} \Delta$, or equivalently $x - x_k \in 2^{-k} \Delta B$.

Observe now that every set $\mathcal{X}^{(j)}$ satisfies $\mathcal{X}^{(j)} \subset 2^{-j+1} \Delta B$. This is the case because every element of $\mathcal{X}^{(j)}$ takes the form $x - \pi_{j-1}(x)$ for some $x \in \mathcal{X}$, and all of these vectors are bounded by $2^{-j+1} \Delta$ in the $\| \cdot \|$ norm by (4.12). Using Lemma 4.2 with the Euclidean norm $\| \cdot \| = \| \cdot \|_2$ and $\Delta = \sqrt{m}$, we get the decomposition $\mathcal{X} \subset \mathcal{X}^{(1)} + \cdots + \mathcal{X}^{(k)} + \frac{\alpha}{2} \sqrt{m} B_2^m$, to which we apply Lemma 3.1. Each algorithm $\mathcal{A}'$ for $1 \leq j \leq k$ is the projection mechanism (as in Lemma 2.2) instantiated with privacy parameter $\rho_j = \rho / k$, and $\mathcal{A}'^{(k+1)}$ (corresponding to the set $\mathcal{X}^{(k+1)} = \frac{\alpha}{2} \sqrt{m} B_2^m$) is the trivial mechanism which always outputs 0.

Let us now analyze the composed mechanism guaranteed by Lemma 3.1. First, observe that this mechanism satisfies $\rho$-zCDP. Indeed, as guaranteed by Lemma 3.1 it satisfies zCDP with privacy parameter $\sum_{j=1}^{k+1} \rho_j = \sum_{j=1}^{k} \frac{\rho}{k} + 0 = \rho$.

We now turn to analysis of the error of the composed mechanism. Observe first that the error of the trivial algorithm $\mathcal{A}'^{(k+1)}$ is bounded as $\text{err}^2(\mathcal{A}'^{(k+1)}, \frac{\alpha}{2} \sqrt{m} B_2^m) \leq \frac{\alpha}{2}$. Let us now consider the error of the mechanism $\mathcal{A}'^{(j)}$, for $1 \leq j \leq k$. The set $\mathcal{X}^{(j)}$ is contained in $2^{-j+1} \sqrt{m} B_2$, and $|\mathcal{X}^{(j)}| \leq \mathcal{P}(\mathcal{X}, 2^{-j})$ because $\mathcal{X}^{(j)}$ is $2^{-j}$-separated — hence we can apply Lemma 2.2 to deduce that for $1 \leq j \leq k$, we have

$$\text{err}^2(\mathcal{A}'^{(j)}, \mathcal{X}^{(j)}) \leq \frac{2^{-j+1} (\log \mathcal{P}(\mathcal{X}, 2^{-j}))^{1/4}}{(2\rho/k)^{1/4} \sqrt{n}}.$$  

(4.11)

We wish to pick $n$ large enough, so that for every $j \leq k$, we have $\text{err}^2(\mathcal{A}'^{(j)}, \mathcal{X}^{(j)}) \leq \frac{\alpha}{2k}$. After rearranging, this will be satisfied as long as for every $j$, we have

$$n \geq \frac{2^{-2j} \sqrt{\log \mathcal{P}(\mathcal{X}, 2^{-j})} / (2\rho/k)^{1/2}}{\sqrt{\rho}}.$$  

(4.12)

Finally, recalling that $k = O(\log \frac{1}{\alpha})$, and the statement of Theorem 1.2 follows.

4.3 The Chaining Mechanism for Worst-Case Error

The proof of Theorem 1.4 is analogous to the previous proof of Theorem 1.2. Here we only give a brief overview of the proof, and defer the details to the appendix.

Our algorithm for worst-case error again applies Lemma 3.1 to a suitable decomposition $\mathcal{X} \subset \mathcal{X}^{(1)} + \cdots + \mathcal{X}^{(k)} + \frac{\alpha}{2} B_\infty^m$, which we get from Lemma 4.2 with the norm $\| \cdot \| = \| \cdot \|_\infty$ and $\Delta = 1$. The basic algorithm we use is the building block to solve each subproblem on universe $\mathcal{X}^{(j)}$ is the Private Multiplicative Weights algorithm (Lemma 2.2) instead of the projection mechanism.

5 Algorithms for Local Differential Privacy

In this section we describe our results in the local model. Once again, we consider the mean point problem, where
each party receives $x_i \in \mathcal{X} \subset \mathbb{R}^m$, which together form a dataset $X$, and the goal is to approximately and privately compute $\hat{X} = \frac{1}{n} \sum_{i=1}^{n} x_i$. We show how to adapt the coarse projection and the chaining mechanisms to the local model and via packing lower bounds show that the sample complexity of our mechanisms is optimal up to constants when the error parameter $\alpha$ is constant.

5.1 Local Projection Mechanism First we discuss a locally private analogue of the projection mechanism. Variants of it appear to have been known to experts, and, independently of our work, Bassily recently published and analyzed a slightly different version \cite{Bas18}. The main observation behind the local projection mechanism is that Gaussian noise addition in the usual projection mechanism can be replaced with a locally private point release mechanism, while the projection step can be performed by the server. The error guarantee of the mechanism — to be described in full shortly — is captured by the following theorem.

**Theorem 5.1.** There exist a non-interactive $\varepsilon$-LDP protocol for the mean point problem $\Pi_{LPM}$, such that for every finite set $\mathcal{X} \subseteq \Delta \sqrt{m} B^m_2$,

$$\text{err}^2(\mathcal{X}, \Pi_{LPM}, n) \lesssim \left( \frac{\Delta g(\mathcal{X})}{\varepsilon \sqrt{nm}} \right)^{1/2} \left( \frac{\Delta^4 \log |\mathcal{X}|}{\varepsilon^2 n} \right)^{1/4}.$$  

Moreover, $\Pi_{LPM}$ has running time polynomial in $n, m$, and $|\mathcal{X}|$.

This error bound should be compared with the bounds for concentrated differential privacy in the centralized model (Lemma \ref{lem:centralized}) — the dependence on the size of the database and the privacy parameter is worse, as the error scales down as $n^{-1/4}$ with the size of the database in local model, as opposed to $n^{-1/2}$ in centralized model.

We will now proceed with a description of $\Pi_{LPM}$ and the proof of this theorem. On input $x_i$, each party $i$ samples a random variable $W_{x_i}$ and sends it to the server. In Section \ref{sec:local_projection}, we describe the distribution of $W_{x_i}$ and prove the following properties of it.

**Lemma 5.1.** For any $x \in \Delta \sqrt{m} B^m_2$ there exists a random variable $W_x$ such that

1. $W_x$ can be sampled in time polynomial in $m$ on input $x$,
2. $\mathbb{E} W_x = x$,
3. $W_x$ is $\sigma$-subgaussian with $\sigma = O(\varepsilon^{-1} \Delta \sqrt{m})$, and
4. for any $x, y \in B^m_2$, we have $D_{\infty}(W_x \| W_y) \leq \varepsilon$.

Assuming this lemma, we will proceed with the proof of Theorem \ref{thm:local_projection}. The protocol is as follows:

1. Each party $i$ samples $W_{x_i}$ and sends it to the server;
2. The server computes the average $\bar{W} := \frac{1}{n} \sum W_{x_i}$ of the responses.
3. The server computes a projection of $\bar{W}$ onto the convex hull of $\mathcal{X}$, that is

$$W^* = \arg \min_{w \in \text{conv}(\mathcal{X})} \|w - \bar{W}\|,$$

and outputs $W^*$.

Note that this mechanism satisfies $\varepsilon$-LDP — the local differential privacy property corresponds directly to the bound on the max-divergence of the variables $W_x$ in item 3 in the statement of Lemma \ref{lem:local_projection}.

The accuracy guarantee follows from the original accuracy analysis of the projection mechanism — the following statement is proved, for instance in \cite{NTZ12, DNT14}. We give the short proof in the appendix for completeness.

**Lemma 5.2.** Consider a convex polytope $K$, and random variable $W$ with $\mathbb{E} W = x \in K$. Take $W^* := \arg \min_{w \in K} \|w - W\|$. Then

$$\mathbb{E} \|W^* - x\|^2 \lesssim \mathbb{E} \sup_{w \in K} \langle w, W - x \rangle.$$  

Notice that if $W$ is a Gaussian centered at $x$, then the right hand side above is $g(K)$. Lemma \ref{lem:projection} then implies that if $W$ is $\sigma$-subgaussian, then the right hand side above is bounded, up to constants, by $\sigma g(K)$. We have the following corollary.

**Corollary 1.** Consider a convex polytope $K$, and $\sigma$-subgaussian random variable $W$ with $\mathbb{E} W = x \in K$. Take $W^* := \arg \min_{w \in K} \|w - W\|$. Then

$$\mathbb{E} \|W^* - x\|^2 \lesssim \sigma g(K).$$

Note that in $\Pi_{LPM}$, as described above, $\mathbb{E} W = \bar{X}$, and moreover $W$ is $\sigma$-subgaussian with $\sigma = O(\frac{\Delta \sqrt{m}}{\varepsilon \sqrt{nm}})$. By applying Corollary \ref{cor:projection} with this $\sigma$, and $K = \text{conv}(\mathcal{X})$, and using our definition of average error

$$\text{err}^2(\mathcal{X}, \Pi_{LPM}, n) = \left( \frac{1}{m} \mathbb{E} \|W^* - \bar{x}\|^2 \right)^{1/2} \lesssim \left( \frac{\Delta g(\mathcal{X})}{\varepsilon \sqrt{nm}} \right)^{1/2}$$

we complete the proof of Theorem \ref{thm:local_projection}.
5.1.1 Local point release In this section we prove Lemma 5.1. We focus on the case \( x \in B_2^m \), i.e. \( \Delta = \frac{1}{\sqrt{m}} \) in the statement of the lemma. The general case follows by scaling \( W_x \) defined below by \( \Delta \sqrt{m} \).

Our construction of \( W_x \) is a variant of the distribution defined in [DJW13] for the mean estimation problem in the local model. We replace their use of the uniform distribution on the sphere with the Gaussian, which is slightly easier to analyze. The definition of \( W_x \) follows.

**Definition 6.** For a given \( x \in B_2^m \), and \( \varepsilon > 0 \), we define a random variable \( W_x \) as follows.

- Take \( U_x \in \{ \pm \frac{x}{\|x\|_2} \} \), such that \( \mathbb{E} U_x = x \).
- Draw \( Z \) at random from a Gaussian distribution \( Z \sim \mathcal{N}(0,1) \).
- Given \( Z \) and \( U_x \), take \( S_x \in \{ \pm 1 \} \) at random, from a distribution such that \( \mathbb{E}[S_x|Z,U_x] = \frac{\varepsilon}{\pi} \text{sign}(\langle Z, U_x \rangle) \).
- Define \( W_x = \frac{3}{\sqrt{\pi} \varepsilon} Z S_x \).

We will first show that the expectation of \( W_x \) is correct — this is an almost direct consequence of the construction.

**Claim 2.** For any \( x \in B_2^m \) and the random variable \( W_x \) defined as in Definition 6 we have \( \mathbb{E} W_x = x \).

**Proof.** If we condition on \( U_x = y_0 \), we have

\[
\mathbb{E} [W_x|U_x = y_0] = \mathbb{E} [(Z, y_0) \cdot \frac{\varepsilon}{3} \text{sign}(\langle Z, y_0 \rangle) ] \cdot \frac{3}{\sqrt{\pi} \varepsilon} = \frac{\varepsilon}{\sqrt{\pi}} \langle Z, y_0 \rangle = \|y_0\|_2 = 1.
\]

On the other hand, for any \( y_1 \perp y_0 \), we have

\[
\mathbb{E} [(W_x, y_1)|U_x = y_0] = \mathbb{E} [(Z, y_0) \cdot \frac{\varepsilon}{3} \text{sign}(\langle Z, y_0 \rangle) ] \cdot \frac{3}{\sqrt{\pi} \varepsilon} = \mathbb{E} [\text{sign}(\langle Z, y_0 \rangle) ] = 0.
\]

because \( \langle Z, y_0 \rangle \) and \( \langle Z, y_1 \rangle \) are independent. Therefore \( \mathbb{E} W_x|U_x = y_0 = y_0 \), and, by the law of total expectation, \( \mathbb{E} W_x = \mathbb{E} U_x \mathbb{E} E[W_x|U_x] = \mathbb{E} U_x = x \).

By a direct application of Fact C.1 to the variables Z and \( S_x \), we obtain the following claim.

**Claim 3.** For any \( x \in B_2^m \), the random variable \( W_x \) is \( O(\varepsilon^{-1}) \)-subgaussian.

We will shift focus to bounding the max divergence between \( W_x \) and \( W_y \). A direct calculation shows the following well-known fact.

**Fact 5.1.** For any \( 0 \leq \varepsilon \leq \frac{1}{2} \) and any two random variables \( S_1, S_2 \in \{ \pm 1 \} \) with \( \|S_1\|, \|S_2\| \leq \varepsilon \), we have \( D_{\infty}(S_1 \| S_2) \leq 3\varepsilon \).

**Claim 4.** For \( x, y \in B_2^m \), we have \( D_{\infty}(W_x \| W_y) \leq \varepsilon \).

**Proof.** We can consider explicitly releasing \( (Z, S_x) \) instead, since they entirely define \( W_x \), and so, by the data-processing inequality, \( D_{\infty}(W_x \| W_y) \leq D_{\infty}((Z, S_x) \| (Z, S_y)) \). For any fixed \( z \), the max-divergence between \( (S_x, Z = z) \) and \( (S_y, Z = z) \) is bounded by \( \varepsilon \) by Fact 5.1 — this is enough to conclude that densities of \( (Z, S_x) \) and \( (Z, S_y) \) are within an \( \varepsilon \) multiplicative factor.

5.2 Local Coarse and Chaining Mechanisms To achieve Theorem 1.5, the local projection mechanism can be lifted to a coarse projection and chaining mechanism via the same general framework as discussed in Section 4.2.

The proof of the first bound for \( A_1 \) in Theorem 1.5 is essentially identical to Theorem 1.1: we consider the same decomposition of the universe \( \mathcal{X} \subset \mathcal{X}^{(1)} \cup \mathcal{X}^{(2)} \), where \( \mathcal{X}^{(1)} \) is a maximal \( \alpha/2 \) separated set, and \( \mathcal{X}^{(2)} := \mathcal{X} \setminus \mathcal{X}^{(1)} \). We apply Lemma 3.1 using a local projection mechanism as the basic algorithm \( A^{(1)} \), and the trivial mechanism that always outputs 0 as the algorithm \( A^{(2)} \). Theorem 5.1 bounds the error of this protocol in terms of \( g(\mathcal{X}^{(1)}) \), and Lemma 4.1 translates this to a bound in terms of covering numbers of \( \mathcal{X} \) as in the theorem statement. The total error of the composed mechanism is bounded as

\[
\text{err}(\mathcal{X}, A_1, n) \leq \left( \frac{(\log \alpha^{-1/2}) H(\mathcal{X}, \alpha/C)^2}{\varepsilon^2 n} \right)^{1/4} \left\{ \varepsilon^2 + \varepsilon \right\}.
\]

therefore if we pick \( n \) as in (1.5), we have \( \text{err}(\mathcal{X}, A_1, n) \leq \alpha \) as desired.

For the construction of the chaining algorithm \( A_2 \) we use instead the decomposition of the set \( \mathcal{X} \) guaranteed by Lemma 4.2. We apply Lemma 3.1 using a local projection mechanism as the basic algorithms \( A^{(1)} \), \ldots, \( A^{(k)} \) instead of the projection mechanism as it was the case in Theorem 1.2 — the dependence of the error of the local projection mechanism on the sample size is quadratically worse, which yields quadratic loss in the sample complexity of the final chaining mechanism, and we lose an additional \( \log(1/\alpha) \) term because of the weaker composition theorem for pure differential privacy.
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References


Proof. [Proof of Claim 1] Consider the \( \ell_2 \) (semi)norm \( \| \cdot \|_{\ell_2} \) defined on random variables \( Z \in \mathbb{R}^m \) by
\[
\|Z\|_{\ell_2} = \E[|Z|^2]^{1/2}.
\]
By the triangle inequality for this norm, we have
\[
\text{err}^2(X, A) = \left( \E[\|A(X) - \bar{X}\|^2] \right)^{1/2} = \left( \E[\|A(X^1) + A(X^2) - \bar{X}^1 - \bar{X}^2\|^2] \right)^{1/2} \leq \left( \E[\|A(X^1) - \bar{X}^1\|^2] \right)^{1/2} + \left( \E[\|A(X^1) - \bar{X}^2\|^2] \right)^{1/2} = \text{err}^2(X^1, A^1) + \text{err}^2(X^2, A^2).
\]

Similarly, \( \text{err}^\infty \) is subadditive by the triangle inequality for (semi)norm \( \| \cdot \|_{\ell_\infty} = \E[|Z|] \) for \( \mathbb{R}^m \)-valued random variables.

Proof. [Proof of Theorem 1.4] We apply Lemma 1.2 with the norm \( \| \cdot \| = \| \cdot \|_\infty \) and \( \Delta = 1 \) to get the decomposition \( X = X^{(1)} + \ldots + X^{(k)} + \frac{\rho}{k} B^m_{\infty} \). We wish to use Lemma 3.1 with this decomposition, to deduce the existence of the mechanism as in the theorem statement. We can use the trivial mechanism which always outputs zero to handle the mean point problem on universe \( \frac{\rho}{k} B^m_{\infty} \) — such a mechanism has error bounded by \( \frac{\rho}{k} \), and satisfies 0-zCDP. For each \( X^{(j)} \), with \( 1 \leq j \leq k \), we will use the Multiplicative Weights Update mechanism (Lemma 2.3) instantiated with privacy parameter \( \rho/k \). Hence, the privacy of composite mechanism is \( \sum_{j=1}^k \rho/k + 0 = \rho \).

The total error of the mechanism guaranteed by

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Lemma 3.1 is bounded by
\[ \sum_{j=1}^{k} C \left( \frac{(\Delta_j (\log |\mathcal{X}(j)|)^{1/4} (\log m)^{1/2} k^{1/4})}{m^{1/4} \sqrt{n}} \right)^{\frac{\alpha}{2}} \]
for some universal constant \( C \).

We wish to pick \( n \) large enough, so that each summand above is bounded by \( \frac{\alpha}{2} \). By rearranging, this is satisfied as soon as
\[ n \geq \sup_{j \leq k} \frac{2^{-2j} \sqrt{\log \mathcal{P}_\alpha(X, 2^{-j})} k^{5/2}}{\sqrt{\rho m^2}}. \]
This is equivalent, up to constant factors, with the following simpler to state condition
\[ n \geq \sup_{t \geq \alpha/4} \frac{t^2 \sqrt{\log \mathcal{P}_\alpha(X, t) (\log \alpha)^{5/2}}}{\sqrt{\rho m^2}}, \]
which completes the proof of the theorem.

Proof. [Proof of Lemma 5.2] Let us write \( W = x + \tilde{W} \) for some mean zero, \( \sigma \)-subgaussian variable \( \tilde{W} \). Note that for any given realization of \( W \), we have \( \|W^* - k\|^2 \leq \langle W, W^* - x \rangle \) — indeed, if this were not the case, then a point on a segment between \( W^* \) and \( x \) would be closer to \( x + \tilde{W} \), and this point belongs to \( K \) by convexity. Therefore
\[ \mathbb{E}[\|W^* - x\|^2] \leq \mathbb{E}[\langle \tilde{W}, W^* - x \rangle] = \mathbb{E}[\langle \tilde{W}, W^* \rangle] - \mathbb{E}[\langle \tilde{W}, x \rangle] = \mathbb{E}[\langle \tilde{W}, W^* \rangle] \leq \mathbb{E}[\sup_{t \in K} \langle \tilde{W}, t \rangle]. \]
where the final inequality follows because \( W^* \in K \) by definition.

B Packing Lower Bounds

B.1 Lower Bounds for CDP The starting point for the proof of Theorem 1.3 is the following theorem bounding the mutual information between the input and the output distribution of a mechanism satisfying CDP.

**Theorem B.1.** (BS16) Let \( \mathcal{A} \) be a mechanism satisfying \( \rho \)-zCDP, and let \( X \) be a random \( n \)-element database. Then,
\[ I(\mathcal{A}(X); X) \leq \log_2(e) m^2. \]

Theorem B.1 and a standard packing argument give our main lemma.

**Lemma B.1.** For any finite \( \mathcal{X} \subseteq [0, 1]^n \) and every \( \alpha, \rho > 0 \), we have
\[ \text{(B.1)} \quad \text{sc}_{\alpha}^2(\mathcal{X}, \alpha) \geq \Omega \left( \frac{1}{\sqrt{\rho}} \cdot \sqrt{\log(\mathcal{P}(\mathcal{X}, 4\alpha))} \right); \]
\[ \text{(B.2)} \quad \text{sc}_{\alpha}^\infty(\mathcal{Q}, \alpha) \geq \Omega \left( \frac{1}{\sqrt{\rho}} \cdot \sqrt{\log(\mathcal{P}_{\infty}(\mathcal{X}, 4\alpha))} \right). \]

Proof. We give the argument for average error; the one for worst-case error proceeds analogously.

Take \( T \) to be a \( 4\alpha \)-separated subset of \( \mathcal{X} \), achieving \( \mathcal{P}(\mathcal{X}, 4\alpha) \). Take \( X \) to be a random database generated by picking a uniformly random element \( V \in T \), and defining \( X \) to be the database consisting of \( n \) copies of \( V \). If \( \mathcal{A} \) is such that \( \text{err}^2(\mathcal{Q}, \mathcal{A}, n) \leq \alpha \), then for any \( X \), by Markov’s inequality we have that, with probability at least \( 3/4 \), \( \|\mathcal{A}(X) - \bar{X}\|_2 \leq 2\alpha^2 \sqrt{m} \). Let \( y \) be the closest point in \( T \) to \( \mathcal{A}(X) \) in Euclidean norm. Since any two elements \( x, x' \) of \( T \) satisfy \( \|x - x'\|_2 > 4\alpha \sqrt{m} \), we have that, with probability at least \( 3/4 \), \( y = x \). By Fano’s inequality,
\[ I(\mathcal{A}(X); X) \geq \frac{3}{4} \log_2(|T| - 1) - 1. \]
The lower bound now follows from Theorem B.1.

Theorem 1.3 is a direct consequence of Lemma B.1 and the following well-known inequality, which follows by a padding argument (see [BÜV14]):
\[ \forall t \geq \alpha : \quad \text{sc}_{\alpha}^2(\mathcal{X}, t) \leq \left[ \frac{\alpha}{t} \right] \cdot \text{sc}_{\alpha}^2(\mathcal{X}, t); \]
\[ \text{sc}_{\alpha}^\infty(\mathcal{X}, t) \leq \left[ \frac{\alpha}{t} \right] \cdot \text{sc}_{\alpha}^\infty(\mathcal{X}, t). \]

B.2 Lower Bound for Local Protocols We prove Theorem 1.6, which shows that the sample complexity of the Local Chaining Mechanism is optimal up to \( O(\alpha^{-2}) \) factor for any \( \mathcal{X} \), whereas optimality is defined over all sequentially interactive protocols satisfying approximate LDP. The proof of this lower bound follows via the framework of Bassily and Smith [BS15], which extends the arguments in [DJW18] to approximate differential privacy. Theorem 1.6 will follow from Fano’s inequality and the following lemma.

**Lemma B.2.** Let \( 0 \leq \beta \leq 1 \), and let \( T \) be a finite subset of \( \mathcal{X} \). For any \( x \in T \) let us define the distribution \( P_x^\beta = \beta 1_x + (1 - \beta) U \), where \( 1_x \) is the point distribution supported on \( x \), and \( U \) is the uniform distribution on \( T \). Let \( V \) be a uniformly random sample from \( T \), and let the elements \( x_1, \ldots, x_n \) of the database \( X \) be sampled independently from \( P_x^\beta \). Then, for any protocol \( \Pi \) satisfying \( (\varepsilon, \delta) \cdot \text{LDP} \),
\[ I(\Pi(X); V) = O \left( \beta^2 \varepsilon^2 n + \frac{\delta}{\varepsilon} n \log |T| + \frac{\delta}{\varepsilon} n \log(\varepsilon/\delta) \right). \]
Lemma B.2 extends Theorem 1 from [DJW18] to approximate LDP. It is also a (straightforward) extension of the lower bound in [BS15] to sequentially interactive mechanisms.

Before we prove Lemma B.2 let us show how it implies Theorem 1.6

**Proof.** [Proof of Theorem 1.6] Take $T$ to be a $t$-separated subset of of $X$ of size $P(X,t)$, for some $t \geq 6\alpha$. Let $\beta = 6\alpha/t$. Observe that for any $x,x' \in T$, and databases $X$ with rows $x_1, \ldots, x_n$ sampled IID from $P_x^\beta$, and $X'$ with rows $x_1, \ldots, x_n$ sampled IID from $P_x^\beta$ we have

$$\frac{1}{\sqrt{m}} \|X - \mathbb{E}X\|_2 \geq \beta t = 6\alpha.$$  

Moreover, by a standard symmetrization argument (see,e.g. Chapter 6 of [LT91]),

$$\mathbb{E} \left( \frac{1}{m} \|X - \mathbb{E}X\|_2^2 \right) \leq 4 \frac{1}{mn} \sum_{i=1}^{n} \|x_i\|_2^2 \leq 4 \frac{n}{n},$$

where we used the fact that $x_i \in [0,1]^m$. An analogous inequality holds for $X'$. By Markov’s inequality, as long as $n \geq \frac{32}{\alpha}$, with probability at least $\frac{1}{2}$ both $X$ and $X'$ are at distance at most $\alpha$ from their expected values, and therefore, we have

$$\frac{1}{\sqrt{m}} \|X - \mathbb{E}X\|_2 \geq 4\alpha.$$  

This means that the output of any protocol $\Pi$ with error $\text{err}^2(X,\Pi) \leq \alpha$ for some $n \geq \frac{32}{\alpha}$, when run on a database $X$ with $n$ rows sampled IID from $P_x^\beta$ for some $x \in T$ allows us to identify $x$ with probability at least $\frac{1}{2}$. Let $V$ be sampled uniformly from $T$ and let the $n$-element database $X$ be sampled IID from $P_x^\beta$. By Fano’s inequality, we have that

$$I(\Pi(X);V) \geq \frac{1}{2} \log(|T| - 1) - 1 = \Omega(\log P(X,t)).$$

Plugging in the upper bound on $I(\Pi(X);V)$ from Lemma B.2 and solving for $n$ finishes the proof.

Towards proving Lemma B.2 we recall two lemmas from [BS15].

**Lemma B.3.** (Claim 5.4 of [BS15]) Let $Z$ be a random variable over some finite set $T$, and let $\mathcal{A}$ be an $(\epsilon, \delta)$-differentially private algorithm defined on one-element databases from $T$. Then,

$$I(\mathcal{A}(Z);Z) = O \left( \epsilon^2 + \frac{\delta}{\epsilon} \log |T| + \frac{\delta}{\epsilon} \log(\epsilon/\delta) \right)$$

Note that above $\mathcal{A}$ being differentially private simply means that for any $t,t' \in T$, it should hold that $D_\infty^\epsilon(\mathcal{A}(t)\|\mathcal{A}(t')) \leq \epsilon$.

We need one final lemma which allows us to get the correct dependence on $\alpha$ in Lemma B.2.

**Lemma B.4.** (Claim 5.5 of [BS15]) Let $\Pi$ be a protocol satisfying $(\epsilon, \delta)$-LDP, and let $\Pi'$ be the protocol over databases in $T^n$ in which the algorithm $\Pi'_i$ run by party $i$ on private input $t_i$ and message $Y_{i-1}$ received from party $i-1$ is defined by:

1. Sample $t'_i$ from $P_{t_i}$;
2. Run $\Pi_{i-1}(t'_i, Y_{i-1})$.

Then $\Pi'$ satisfies $(O(\epsilon \alpha), O(\alpha \epsilon))$-LDP.

**Proof.** [Proof of B.2] Observe first that running protocol $\Pi$ on $X$ is equivalent to running the protocol $\Pi'$ defined in Lemma B.4 on the random database consisting of $n$ copies of $V$. Therefore,

$$I(\Pi(V);V) = I(\Pi'(V);V) = I(\Pi'_1, \ldots, \Pi'_n;V),$$

where we use $\Pi, \Pi', \Pi_i$ as shorthands for $\Pi(X)$, $\Pi(V,Y_{i-1})$, and $\Pi'_i(V,Y_{i-1})$, respectively, and $Y_{i-1}$ is the message received by party $i$ from party $i-1$, defined to be empty for $i = 1$. By the chain rule of mutual information,

$$I(\Pi'(V);V) = I(\Pi'_1;V) + I(\Pi'_2;V | \Pi'_1) + \ldots + I(\Pi'_n;V | \Pi'_1, \ldots, \Pi'_{n-1}).$$

By Definition 3 and Lemma B.4, conditional on $\Pi'_i = (y_i, z_i), \ldots, \Pi'_{i-1} = (y_{i-1}, z_{i-1})$ for any possible views $y_1, \ldots, y_{i-1}$ and $z_1, \ldots, z_{i-1}$, the algorithm $\Pi'_i(y_i, z_{i-1})$ satisfies $(O(\epsilon \alpha), O(\alpha \epsilon))$-differential privacy. Then by Lemma B.3

$$I(\Pi'_i;V | \Pi'_i = (y_i, z_i), \ldots, \Pi'_{i-1} = (y_{i-1}, z_{y-1})) = I(\Pi'_i(V, y_{i-1});V) = O \left( \alpha^2 \epsilon^2 + \frac{\delta}{\epsilon} \log |T| + \frac{\delta}{\epsilon} \log(\epsilon/\delta) \right).$$

Taking expectation over $\Pi'_i, \ldots, \Pi'_{i-1}$ and summing up over $i$, we get

$$I(\Pi(V);V) = I(\Pi';V) = O \left( \alpha^2 \epsilon^2 n + \frac{\delta}{\epsilon} n \log |T| + \frac{\delta}{\epsilon} n \log(\epsilon/\delta) \right).$$

This finishes the proof of the lemma.
C Basic facts about subgaussian random variables

The following lemma characterizes subgaussian random variables in terms of their moments. See Chapter 2 of [BLM13] for a proof.

Lemma C.1. There exist universal constants $C_1, C_2$ such that

- If $W$ is a mean zero, $\sigma$-subgaussian random variable, then for every even $p$, and every $z \in \mathbb{R}^m$ we have $\|\langle W, z \rangle\|_p \leq C_1 \sigma \sqrt{p} \|z\|$.

- If for every even $p$ and every $z \in \mathbb{R}^m$, we have $\|\langle W, z \rangle\|_p \leq C_2 \sigma \sqrt{p} \|z\|$, then $W$ is $\sigma$-subgaussian.

The following lemma bounds the parameter of the sum of independent subgaussian random variables. Once again, the proof can be found in Chapter 2 of [BLM13].

Lemma C.2. If $W_1, W_2, \ldots, W_k$ are $\sigma_i$-subgaussian independent random variables, then $\sum_{i \leq k} W_i$ is $\sqrt{\sigma_1^2 + \ldots + \sigma_k^2}$-subgaussian.

The following deep fact is a consequences of Talagrand’s majorizing measures theorem [Tal87] (see also the book [Tal14]).

Lemma C.3. For a $\sigma$-subgaussian random variable $W \in \mathbb{R}^m$, and arbitrary set $K \subset \mathbb{R}^m$, we have $\mathbb{E} \sup_{k \in K} \langle W, k \rangle \lesssim \sigma g(K)$.

Finally we prove the following simple fact.

Fact C.1. Consider a pair of random variables $(Z, S)$ such that $Z \in \mathbb{R}^m$ is mean zero, $\sigma$-subgaussian random variable, and $S \in \{\pm 1\}$, not necessarily independent from $Z$. Then $SZ$ is $O(\sigma)$ subgaussian.

Proof. By Lemma C.1 is enough to check that for every $p$ and any $v \in \mathbb{R}^m$, we have $\|\langle SZ, v \rangle\|_p \lesssim \sigma \sqrt{p}$. But we have $\|\langle SZ, v \rangle\|_p \leq |||S| \cdot \langle Z, w \rangle\|_p = \|\langle Z, w \rangle\|_p \leq \sigma \sqrt{p}$. 

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